Kennedy Baptist College

Semester One Examination, 2023

Question/Answer booklet

MATHEMATICS SPECIALIST UNIT 3

Section One: Calculator-free

SOLUTIONS

WA student number: In figures | | | | | | | | |

In words _____

_____

Your name _____

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Number of additional answer booklets used (if applicable): | |

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	7	7	50	48	35
Section Two: Calculator-assumed	12	12	100	90	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

See next page

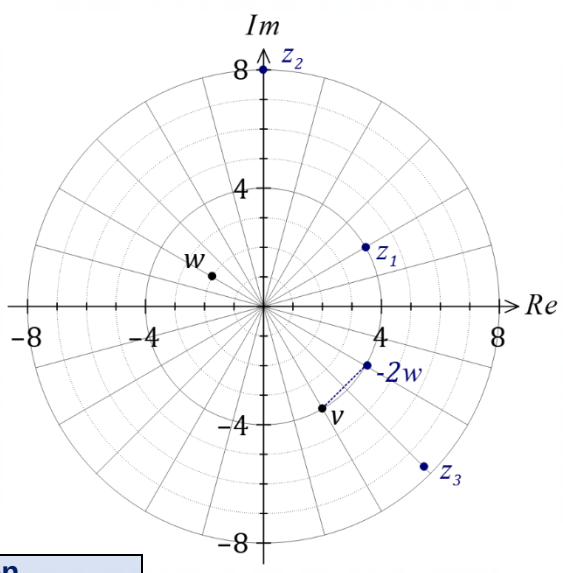
Section One: Calculator-free **35% (48 Marks)**

This section has **seven** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(7 marks)**

The diagram shows the complex numbers v and w in the Argand plane.



(a) Express v in

(i) polar form. (1 mark)

Solution
$v = 4\,\text{cis}\left(-\frac{\pi}{3}\right)$
Specific behaviours
✓ correct expression

(ii) Cartesian form. (1 mark)

Solution
$v = 2 - 2\sqrt{3}i$
Specific behaviours
✓ correct expression

(b) Plot and label the following complex numbers on the diagram above:

(i) $z_1 = iv$. (1 mark)

Solution
Rotates v by 90° about O.
Specific behaviours
✓ plots correctly

(ii) $z_2 = vw$. (2 marks)

Solution
$\lvert vw\rvert = 2 \times 4 = 8, \quad \arg(vw) = -\frac{\pi}{3} + \frac{5\pi}{6} = \frac{\pi}{2}$
Specific behaviours
✓ correct argument ✓ correct modulus

(iii) $z_3 = v - 2w$. (2 marks)

Solution
Use vector addition of v and $-2w$, so that $\arg(v - 2w) = -\frac{\pi}{4}, \quad \lvert v - 2w\rvert \approx 7.7$
Specific behaviours
✓ correct argument ✓ correct modulus

See next page

Question 2 **(6 marks)**

The Cartesian equations for three planes are $x - y - z = 2,\ 2x - y + z = 7$ and $3x + y + z = 2$.

(a) Show that none of these planes is parallel to another. (2 marks)

Solution
The planes have normal vectors $\begin{pmatrix}1\\-1\\-1\end{pmatrix}, \begin{pmatrix}2\\-1\\1\end{pmatrix}, \begin{pmatrix}3\\1\\1\end{pmatrix}$ and since none of these are scalar multiples of each other, then none of the three planes is parallel to one of the others.
Specific behaviours
✓ correctly states all normal vectors ✓ correct explanation

(b) Solve the three equations simultaneously. (3 marks)

Solution
$x - y - z = 2$ $3x + y + z = 2$ $4x = 4, \quad x = 1$ $1 - y - z = 2$ $2 - y + z = 7$ $3 - 2y = 9, \quad y = -3$ $3 - 3 + z = 2, \quad z = 2$ $x = 1, \quad y = -3, \quad z = 2$
Specific behaviours
✓ uses elimination to obtain value of x ✓ uses elimination to obtain a second value ✓ states correct solution set

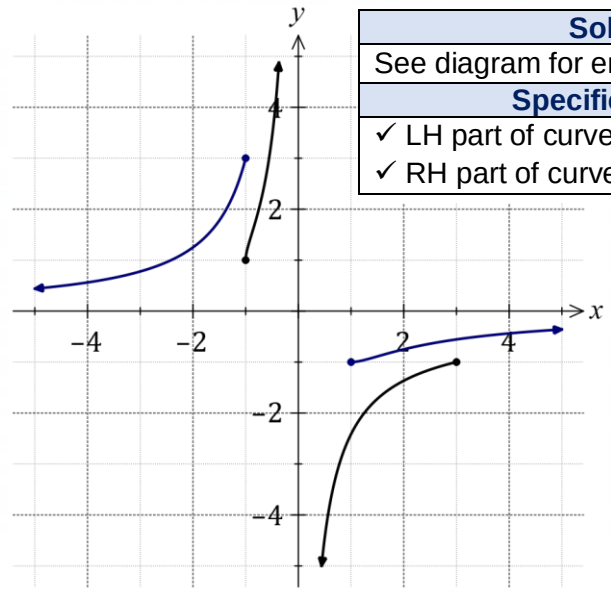
(c) State the geometric interpretation of the solution obtained in part (b). (1 mark)

Solution
Three non-parallel planes intersecting at the point $(1, -3, 2)$.
Specific behaviours
✓ correctly interprets solution

See next page

Question 3 **(7 marks)**

The diagram shows the graph of $y = f(x)$, where $f(x) = \dfrac{1}{1-\sqrt{x+1}}$ and the domain of f is restricted to $\{x \in \mathbb{R} \mid -1 \leq x \leq 3,\ x \neq 0\}$.



Solution (b)
See diagram for endpoints and curvature
Specific behaviours
✓ LH part of curve from $(-1,3)$
✓ RH part of curve from $(1,-1)$

(a) Explain how to use the graph to estimate a solution to the equation $f^{-1}(x) = 2$. (1 mark)

Solution
Draw the vertical line $x = 2$ and the y-coordinate of the intersection of this line and the curve will be the solution. *(Do not accept use of graph of inverse function)*
Specific behaviours
✓ correct explanation

(b) On the same axes, sketch the graph of $y = f^{-1}(x)$. (2 marks)

(c) Determine a simplified rule for $y = f^{-1}(x)$, stating any domain restriction(s). (4 marks)

Solution
Range of f, $y \leq -1 \cup y \geq 1$, is domain of f^{-1}.
$x = \dfrac{1}{1-\sqrt{y+1}}$
$-\sqrt{y+1} = \dfrac{1}{x} - 1$
$y + 1 = \dfrac{1}{x^2} - \dfrac{2}{x} + 1$
$y = f^{-1}(x) = \dfrac{1}{x^2} - \dfrac{2}{x}, \quad \{x \in \mathbb{R} \mid x \leq -1 \cup x \geq 1\}$
Specific behaviours
✓ interchanges x, y and cross multiplies
✓ obtains expression for $\sqrt{y+1}$
✓ obtains defining rule for inverse
✓ states domain restrictions in terms of x for inverse

See next page

Question 4 **(7 marks)**

The coordinates of three points in space are $L(0,3,3), M(-2,1,-1)$ and $N(-1,1,2)$.

(a) Determine the vector equation of the sphere with diameter LM. (3 marks)

Solution

Centre:

$$\frac{1}{2}\left(\begin{pmatrix}0\\3\\3\end{pmatrix}+\begin{pmatrix}-2\\1\\-1\end{pmatrix}\right)=\begin{pmatrix}-1\\2\\1\end{pmatrix}$$

Radius:

$$\left|\begin{pmatrix}0\\3\\3\end{pmatrix}-\begin{pmatrix}-1\\2\\1\end{pmatrix}\right|=\sqrt{1^2+1^2+2^2}=\sqrt{6}$$

Equation:

$$\left|\underset{\sim}{r}-\begin{pmatrix}-1\\2\\1\end{pmatrix}\right|=\sqrt{6}$$

Specific behaviours

✓ calculates centre
✓ calculates radius or diameter
✓ correct vector equation

(b) Determine the Cartesian equation of the plane that contains all three points. (4 marks)

Solution

$$\overrightarrow{ML}=\begin{pmatrix}0\\3\\3\end{pmatrix}-\begin{pmatrix}-2\\1\\-1\end{pmatrix}=\begin{pmatrix}2\\2\\4\end{pmatrix},\qquad \overrightarrow{NL}=\begin{pmatrix}0\\3\\3\end{pmatrix}-\begin{pmatrix}-1\\1\\2\end{pmatrix}=\begin{pmatrix}1\\2\\1\end{pmatrix}$$

Normal to plane:

$$\underset{\sim}{n}=\begin{pmatrix}1\\1\\2\end{pmatrix}\times\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}-3\\1\\1\end{pmatrix}$$

Constant:

$$\begin{pmatrix}-3\\1\\1\end{pmatrix}\cdot\begin{pmatrix}0\\3\\3\end{pmatrix}=6$$

Cartesian equation:

$$-3x+y+z=6$$

Specific behaviours

✓ derives two vectors in the plane
✓ calculates normal to plane
✓ calculates constant
✓ correct cartesian equation

See next page

Question 5 **(7 marks)**

Consider the function $f(z) = z^4 + 4z^3 + 10z^2 + 20z + 25$.

(a) Determine the remainder when $f(z)$ is divided by $z - i$. (1 mark)

Solution

$$\begin{aligned} f(i) &= i^4 + 4i^3 + 10i^2 + 20i + 25 \\ &= 1 - 4i - 10 + 20i + 25 \\ &= 16 + 16i \end{aligned}$$

Specific behaviours

✓ correct remainder

(b) Show that $z - \sqrt{5}i$ is a factor of f. (2 marks)

Solution

$$\begin{aligned} f\left(\sqrt{5}i\right) &= \left(\sqrt{5}i\right)^4 + 4\left(\sqrt{5}i\right)^3 + 10\left(\sqrt{5}i\right)^2 + 20\sqrt{5}i + 25 \\ &= 25 + 4\left(-5\sqrt{5}\right) + 10(-5) + 20\sqrt{5}i + 25 \\ &= 25 - 20\sqrt{5}i - 50 + 20\sqrt{5}i + 25 \\ &= 0 \end{aligned}$$

Specific behaviours

✓ correctly evaluates powers of $\sqrt{5}i$
✓ simplifies to show line that clearly sums to zero

(c) Solve $f(z) = 0$. (4 marks)

Solution

Since $z - \sqrt{5}i$ is a factor then $z + \sqrt{5}i$ must also be a factor.

$$\begin{aligned} z^4 + 4z^3 + 10z^2 + 20z + 25 &= \left(z + \sqrt{5}i\right)\left(z - \sqrt{5}i\right)q(z) \\ &= (z^2 + 5)q(z) \\ &= (z^2 + 5)(z^2 + 4z + 5) \end{aligned}$$

$$\begin{aligned} z^2 + 4z + 5 &= 0 \\ (z + 2)^2 - 4 &= -5 \\ (z + 2)^2 &= -1 = i^2 \\ z + 2 &= \pm i \\ z &= -2 \pm i \end{aligned}$$

Hence $f(z) = 0$ when $z = \pm\sqrt{5}i,\ z = -2 \pm i$.

Specific behaviours

✓ uses complex conjugate to obtain one quadratic factor of $f(z)$
✓ determines second quadratic factor $q(z)$
✓ shows use of appropriate method to solve $q(z) = 0$
✓ states all solutions

See next page

Question 6 **(7 marks)**

(a) Given that $w = \dfrac{\sqrt{3} - i}{1 + i}$, determine the modulus and argument of w. (3 marks)

Solution

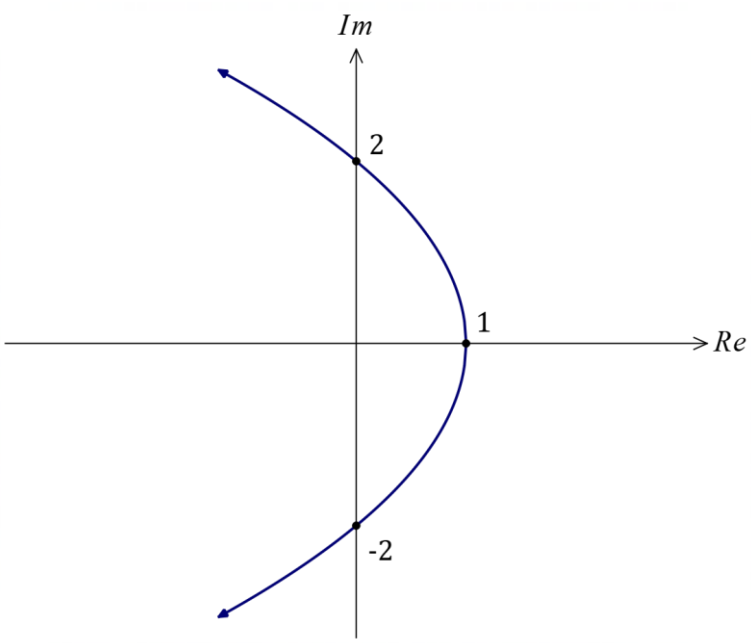
$$u = \sqrt{3} - i = 2\,\text{cis}\left(-\frac{\pi}{6}\right), \qquad v = 1 + i = \sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$$

$$|w| = \frac{2}{\sqrt{2}} = \sqrt{2}, \qquad \arg w = \arg u - \arg v = -\frac{\pi}{6} - \frac{\pi}{4} = -\frac{5\pi}{12}$$

Specific behaviours

✓ expresses numerator and denominator in polar form
✓ modulus
✓ argument

(b) Sketch the subset of the complex plane determined by $-2|z| = z + \overline{z} - 4$. (4 marks)



Solution

Let $z = x + iy$ so that

$$-2|x + iy| = x + iy + x - iy - 4$$
$$|x + iy| = 2 - x$$
$$x^2 + y^2 = 4 - 4x + x^2$$
$$y^2 = 4 - 4x$$
$$x = 1 - \frac{y^2}{4}$$

Specific behaviours

✓ uses Cartesian form to eliminate i
✓ obtains relationship
✓ sketches parabolic curve
✓ correct vertex and other intercepts

See next page

Question 7 **(7 marks)**

Consider functions $f(x) = \dfrac{x^2 + 7}{2}$ and $g(x) = \sqrt{25 - x^2}$.

(a) Explain why f is not a one-to-one function. (1 mark)

Solution
f is a many-to-one function. For example, $f(1) = f(-1) = 4$.
Specific behaviours
✓ states many-to-one or uses examples to show not one-to-one

(b) State the domain and range of $g(x)$. (2 marks)

Solution
$D_g: -5 \leq x \leq 5, \quad R_g: 0 \leq y \leq 5.$
Specific behaviours
✓ correct domain ✓ correct range

(c) Determine the domain and range of $g\big(f(x)\big)$. (4 marks)

Solution
$g\big(f(x)\big) = \sqrt{25 - f(x)^2}$ Using result from (b) we require $-5 \leq f(x) \leq 5$ but since the natural range of f is $y \geq \dfrac{7}{2}$ then for domain of $g \circ f$ we just need the restriction $f(x) \leq 5$: $\dfrac{x^2 + 7}{2} \leq 5 \Rightarrow x^2 \leq 3 \Rightarrow D_{g\circ f}: -\sqrt{3} \leq x \leq \sqrt{3}$ Use $R_f = \left\{\dfrac{7}{2} \leq y \leq 5\right\}$ to obtain range of $g \circ f$: $g\left(\dfrac{7}{2}\right) = \sqrt{25 - {}^{49}/_{4}} = \dfrac{\sqrt{51}}{2}, \quad g(5) = 0 \Rightarrow R_{g\circ f}: 0 \leq y \leq \dfrac{\sqrt{51}}{2}$
Specific behaviours
✓ indicates that $f(x) \leq 5$ ✓ correct domain ✓ indicates restricted range of f ✓ correct range

End of questions

Supplementary page

Question number: _____

Supplementary page

Question number: _____